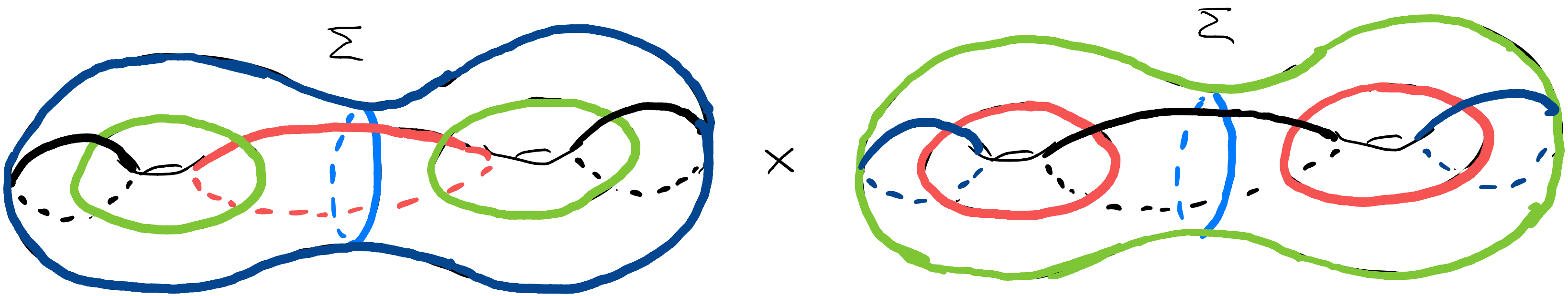
Higher-dimensional hyperbolic manifolds via Coxeter polytopes

Ventotene 2025

A hyperbolic link of 7 tori in a product of two surfaces:



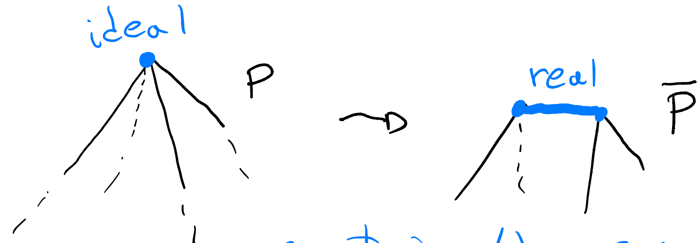
$M^4 = (\Sigma \times \Sigma) \setminus (T_1 \sqcup \cdots \sqcup T_7)$ is hyperbolic with 7 cusps

$\chi(M^4) = \chi(\Sigma \times \Sigma) = (-2)\cdot(-2) = 4$

A SANITY CHECK: Must be $\|\Sigma \times \Sigma\| \leq \|M^4\|$ [Fujiwara-Manning]

- $\|\Sigma \times \Sigma\| = 6\,\chi(\Sigma \times \Sigma) = 6 \cdot 4 = \underline{24}$ [Bucher]
- $\|M^4\| = \dfrac{\mathrm{Vol}(M^4)}{v_4} = \dfrac{4\pi^2}{3v_4}\chi(M) \sim 49\,\chi(M) = 49 \cdot 4 = \underline{196}$

Proof:



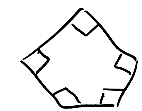
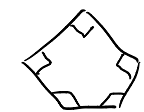
mantain the same colouring

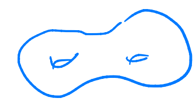
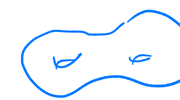
P^4 has 10 facets

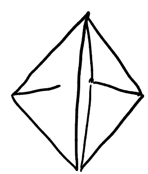
5 ideal vertices

5 real vertices

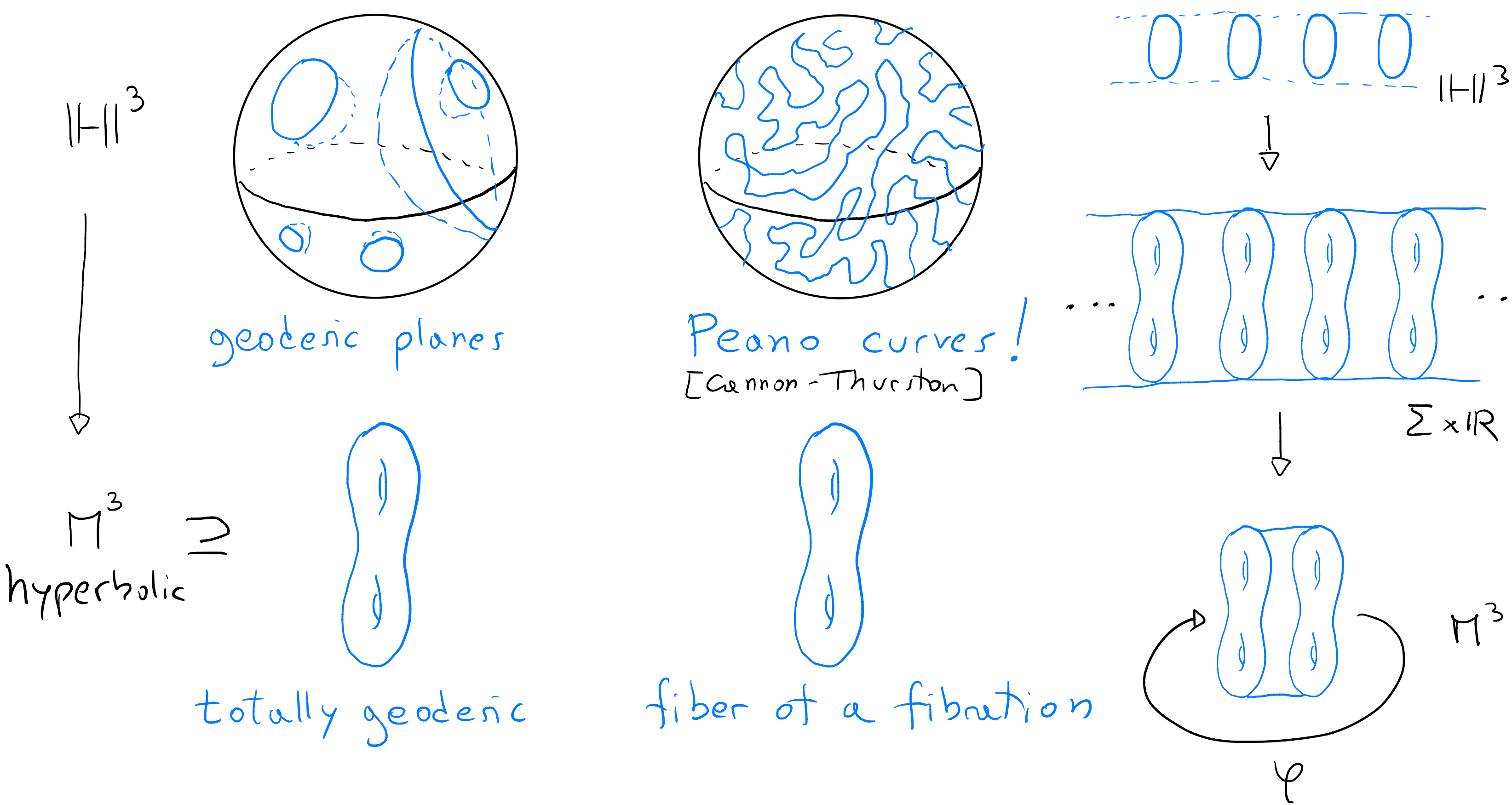
M^4

$\bar{P}^4 =$  $\times$ 

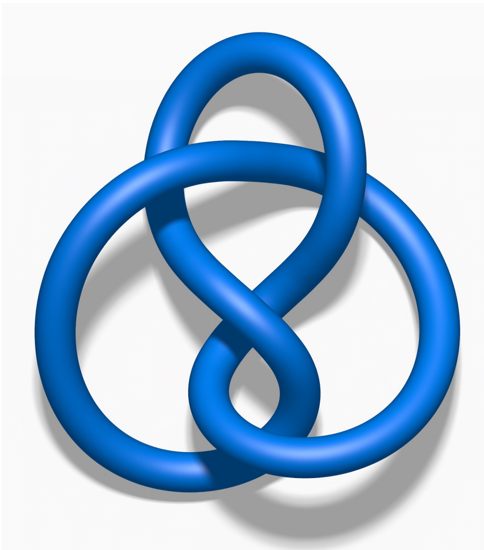
$\bar{M}^4 =$  $\times$ 

$\bar{P}^4 =$  $\rightsquigarrow \bar{M}^4 = S^4$

A fibration on a hyperbolic manifold is quite paradoxical:



A very small example in dimension 3

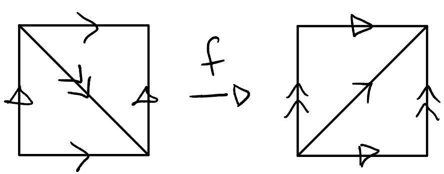


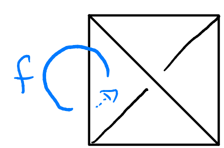
The figure-8 knot complement double covers the Gieseking manifold that is constructed as follows:

$$A = \begin{pmatrix}1&1\\1&0\end{pmatrix} \qquad A^2 = \begin{pmatrix}2&1\\1&1\end{pmatrix} \qquad \mathbb{R}^2/\mathbb{Z}^2$$

$$f\begin{pmatrix}x\\y\end{pmatrix} = A\begin{pmatrix}x\\y\end{pmatrix} \text{ descends to } f: \underset{\mathbb{R}^2/\mathbb{Z}^2}{\overset{\parallel}{T^2}} \circlearrowleft$$

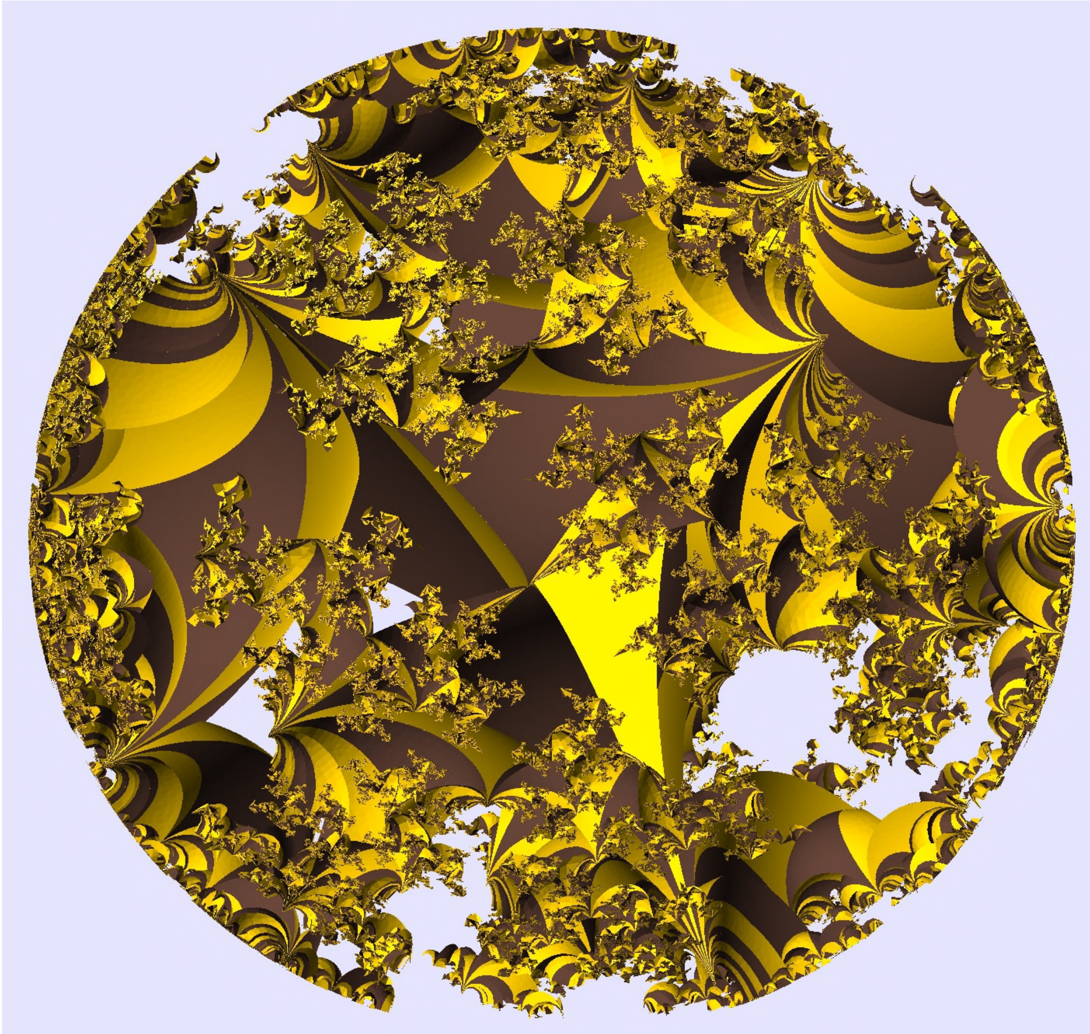
and fixes $0 \in T^2$, hence preserves $T^2_0 = T^2 \setminus \{0\}$





ideal regular hyperbolic tetrahedron

The Gieseking manifold is $\underset{\text{MAPPING TORUS}}{M^3} = T^2_0 \times [0,1]/\sim \qquad (x,1) \sim (f(x),0)$



picture by
D. Calegari

In general dimension n?

Chern-Gauss-Bonnet: n even $\Longrightarrow \chi(M) \neq 0 \quad \forall M$ hyperbolic

$\Downarrow$

Circle-valued Morse function: $M \xrightarrow{f} S^1$ $\qquad$ M does not fiber

We have $\boxed{\chi(M) = \sum_{i=0}^{n} (-1)^i c_i}$

$c_i = \#\{\text{critical points with index } i\}$

f is PERFECT if $|\chi(M)| = \sum_{i=0}^{n} c_i$

when n is odd: $\quad f$ perfect $\Longleftrightarrow f$ fibration

When you have one, you have many:

- fibrations (perfect functions) lift to finite covers;
- fibrations (perfect functions) can be perturbed, i.e.

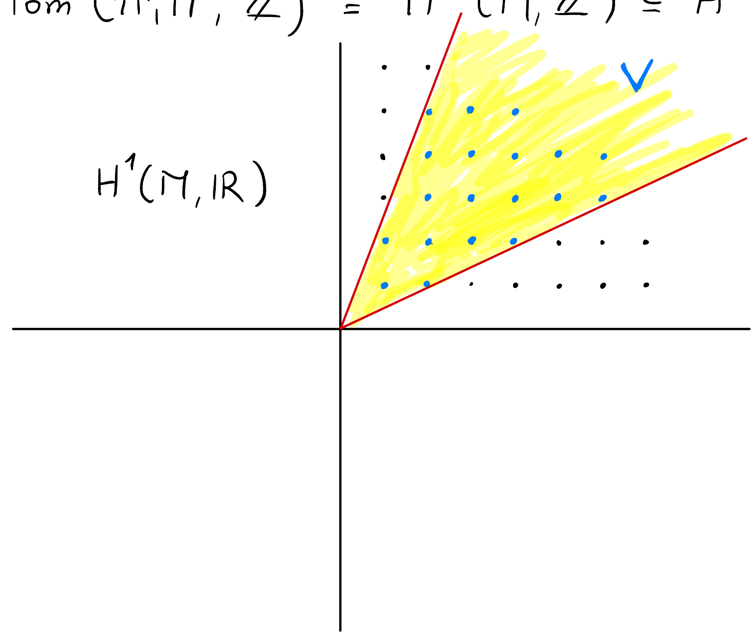
$$M \xrightarrow{f} S^1 \dashrightarrow [f] \in [M, S^1] = \mathrm{Hom}(\pi_1 M, \mathbb{Z}) = H^1(M, \mathbb{Z}) \subseteq H^1(M, \mathbb{R})$$

Fibrations realise

$V \cap H^1(M, \mathbb{Z})$

with $V \subseteq H^1(M, \mathbb{R})$

open cone

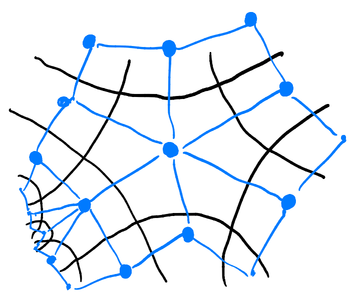


[Battista, M.] [Italiano, M., Migliorini] [Italiano, Migliorini]

Thm: The manifolds M^3, M^4, M^5, M^6 have a perfect circle-valued Morse function

(that is a fibration in odd dimension)

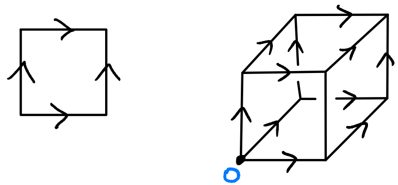
proof: $M = \{P_v\} \rightsquigarrow$ Dual cube complex C



When M has cusps it collapses onto C

Bestvina - Brady theory

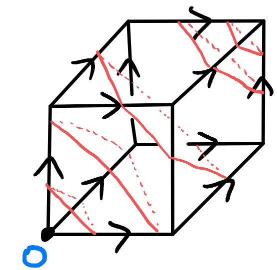
COHERENT ORIENTATION
ON EDGES



DIAGONAL MAP

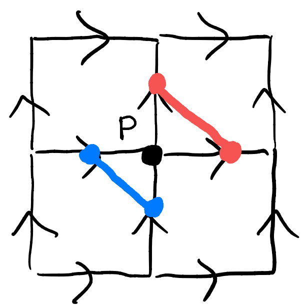
$$f : [0,1]^k \longrightarrow \mathbb{R}/\mathbb{Z}$$

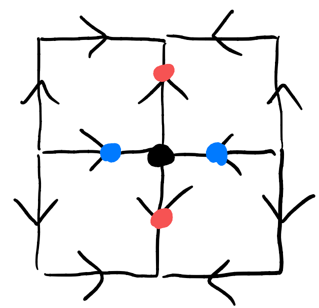
$$x \longmapsto x_1 + \cdots + x_n$$



We get a PL map $f : C \longrightarrow \mathbb{R}/\mathbb{Z}$ that is regular everywhere except possibly at vertices

Ascending and descending links





Lemma: If they collapse to points, P is regular for f

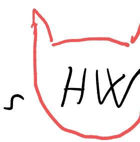
" " " " spheres, P is Morse critical for f

A very small example in dimension 5

The Ratcliffe-Tschantz 5-manifold N^5 has smallest known volume $\frac{7}{4}\zeta(3) = 2.103\ldots$ and two cusps

$$H_1 = \mathbb{Z} \times \mathbb{Z}/_{4\mathbb{Z}} \qquad H_2 = \left(\mathbb{Z}/_{4\mathbb{Z}}\right)^2 \qquad H_3 = \mathbb{Z} \qquad H_4 = \mathbb{Z}$$

It fibers over the circle [Italiano, M., Migliorini]

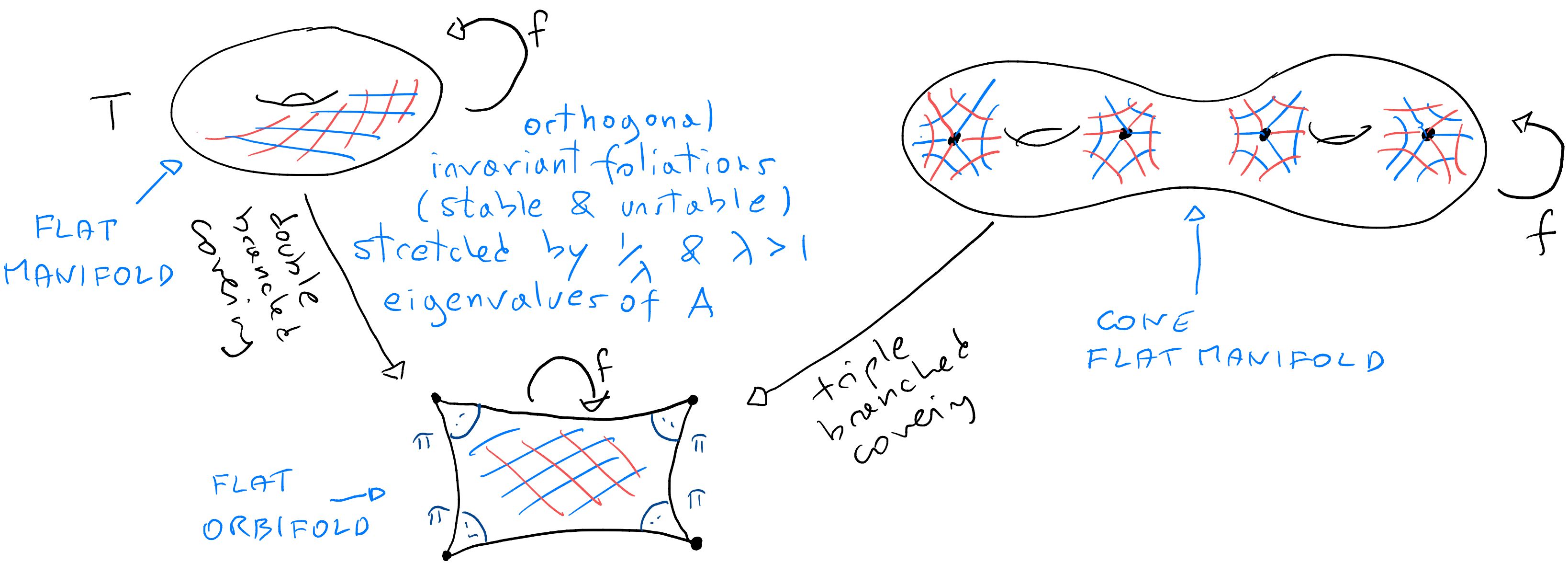
The fiber F^4 is an aspherical 4-manifold with 5 boundary components 

$$H_1 = \left(\mathbb{Z}/_{4\mathbb{Z}}\right)^4 \qquad H_2 = \mathbb{Z}^4 \qquad H_3 = \mathbb{Z}^4 \qquad \chi(F^4) = 1$$

An inspiring construction

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ descends to $f : T^2 \to T^2$

$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$ $\quad O^2 = T^2 / \langle g \rangle$ $\quad f$ descends to $f : O^2 \to O^2$



A direct construction of F^4 and the monodromy $\varphi: F^4 \to F^4$

(such that $N^5 = F^4 \times [0,1]/\sim$ $\quad (x,1) \sim (\varphi(x), 0)$ is the Ratcliffe-Tschantz hyp 5-mfd)

$$\zeta = e^{\frac{2\pi i}{5}} \qquad \begin{array}{l} \mathbb{Z}[\zeta] \xhookrightarrow{\ id\ } \mathbb{C} \\ \mathbb{Z}[\zeta] \hookrightarrow \mathbb{C}, \quad \zeta \mapsto \zeta^2 \end{array} \quad \Rightarrow \quad \mathbb{Z}[\zeta] \xhookrightarrow{\ f\ } \mathbb{C} \times \mathbb{C}, \quad \zeta \longmapsto (\zeta, \zeta^2)$$

$\Lambda = \mathrm{Im}(f)$ is a lattice in $\mathbb{C}^2$ with basis

$$(1,1) \quad (\zeta, \zeta^2) \quad (\zeta^2, \zeta^4) \quad (\zeta^3, \zeta)$$

$T^4 = \mathbb{C}^2/\Lambda$ is a 4-torus with many automorphisms

Every group automorphism of $\mathbb{Z}[\zeta]$ yields an automorphism of T^4

$r(z) = \zeta z$ yields $r(z,w) = (\zeta z, \zeta^2 w)$ isometry

$s(z) = -\bar{z}$ " $s(z,w) = (-\bar{z}, -\bar{w})$ isometry

$\varphi(z) = \lambda z$ " $\varphi(z,w) = (\lambda z, -\lambda^{-1} w)$ affine, ∞ order

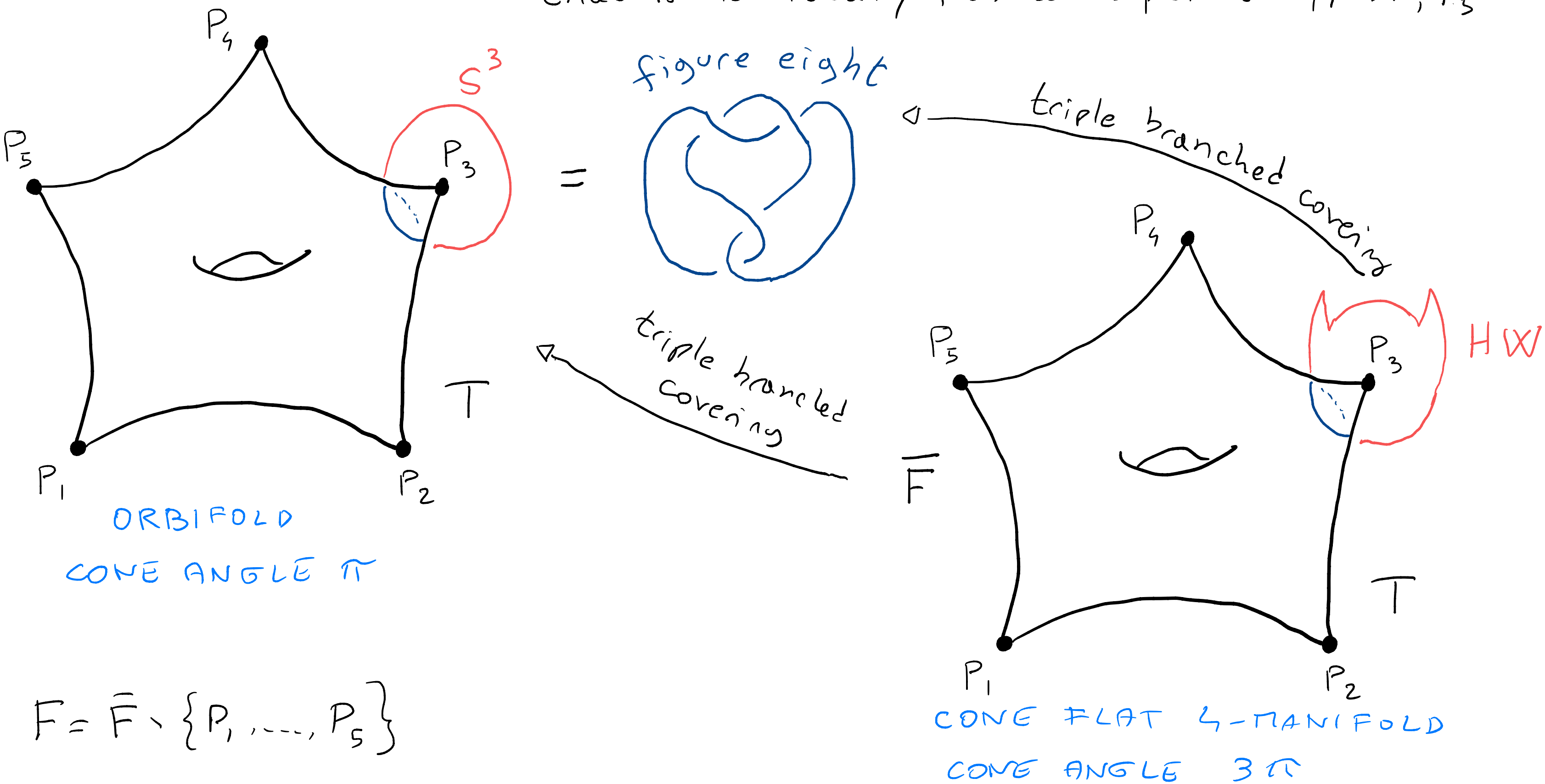
$\lambda = \frac{\sqrt{5}+1}{2} = -\zeta^2 - \zeta^3$ GOLDEN RATIO $\lambda^{-1} = \zeta + \zeta^4 = \frac{\sqrt{5}-1}{2}$

$\mathbb{C} \times \{pt\}$ & $\{pt\} \times \mathbb{C}$ yield orthogonal UNSTABLE and STABLE invariant geodesic foliations of dimension 2 in T^4

$D_{10} = \langle r, s \rangle \curvearrowright T^4$ $O^4 = T^4 / D_{10}$

f descends to $f : O^4 \to O^4$

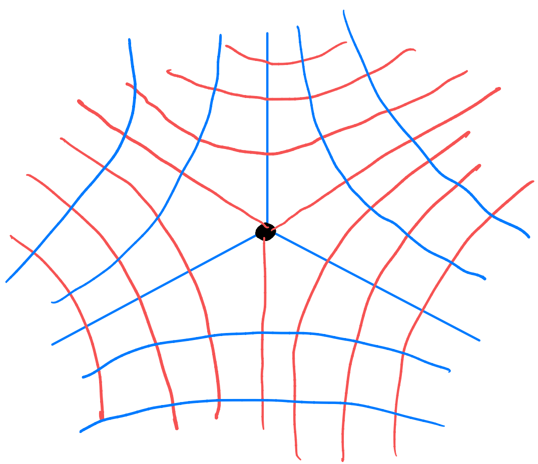
What is O^4? It is S^4 with singular set a torus $T \subseteq S^4$ that is not locally flat at 5 points $P_1, \dots, P_5$



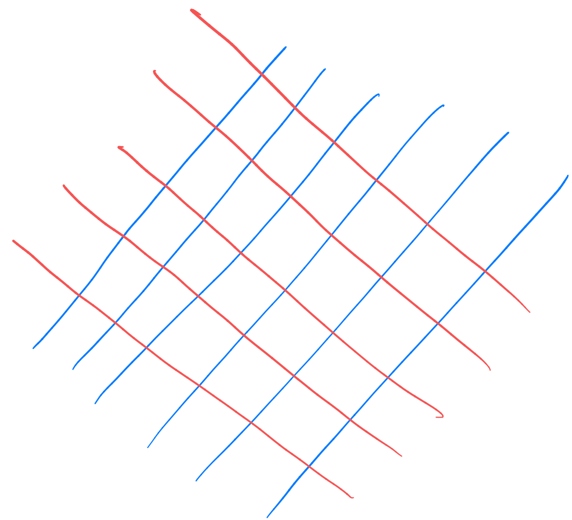
$$F = \bar{F} \setminus \{P_1, \dots, P_5\}$$

The induced $\varphi: \bar{F} \to \bar{F}$ has two orthogonal geodesic foliations that are expanded by λ, $1/\lambda$

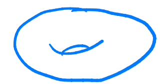
They look like a product

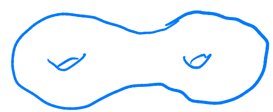


$\times$



Cor: $\exists$ manifold M^4 such that

1) no element in $H_2(M)$ is represented by immersed  or 

2) ∞ many elements in $H_2(M)$ are represented by embedded 

After passing to sufficiently large finite cover:

$$\bar{M} = \bar{F} \times [0,1] /_{\sim} \qquad (x,1) \sim (\varphi(x), 0)$$

[Fujiwara-Manning] : $\pi_1(\bar{M})$ is Gromov hyperbolic

<u>Thm</u> [Italiano, M., Migliorini]

There exists a finite type $H < G$ in hyperbolic G which is not hyperbolic

$G = \pi_1(\bar{M})$, $H = \pi_1(\bar{F})$

<u>Conj</u>: $\bar{F}$ is locally CAT(0)

↳ <u>Cor</u>: $\exists \underset{\text{cpt}}{X}$ locally CAT(0), $\pi_1(X)$ not hyperbolic, $\pi_1(X) \not\supseteq \mathbb{Z} \times \mathbb{Z}$